We consider a periodically driven quantum system in interaction with a phonon heat-bath, which after a sufficiently long time reaches a “steady-state” periodic with the driving frequency. In the simple model of a periodically driven harmonic oscillator with damping we find that the steady-state density operator $\rho_{(st)}$ has no diagonal form in the Floquet-representation. However, for this model it is possible to construct explicitly a basis of states depending on the strength of the damping which diagonalize $\rho_{(st)}$ and lead to a thermal distribution in the steady-state. This basis may be considered as a generalized Floquet-basis for the dissipative system.
1. Introduction

Quantum systems, which are on the one hand in contact with a heat bath and driven by an external periodic force on the other hand, have been the subject of investigation in various physical contexts. In quantum optics, such a system is studied e.g. in resonance fluorescence where a beam of atoms interacts with a coherent laser field and all the electromagnetic modes of the vacuum \[1\], \[2\]. The kicked rotor with dissipation stands for a representative of quantum chaos \[3\], and a small metal ring driven by a time-dependent magnetic flux in interaction with phonons \[4\] is a standard example of a mesoscopic system. In these models dissipation has been realized via the coupling of the system’s degrees of freedom to a bath of harmonic oscillators, which is thought to stay at thermal equilibrium during the interaction process. After tracing over the bath’s variables the dynamics of the system follows from a master equation for the density operator \(\rho(t)\) \[5\], \[6\]. Usually this master equation is derived by the following scheme: Let

\[
H = H_0 + H_B + H_I + H_F
\]

be the Hamiltonian of the system plus bath: \(H_0, H_B\) represent the system and the bath alone, \(H_I\) describes the interaction between system and bath,
furtheron assumed to be weak. $H_F = -xF(t)$ stands for the external driving by a force $F(t)$, which is periodic in time with a period $T$: $F(t + T) = F(t)$. The motion of the density operator $\rho(t)$ is determined by the von Neumann equation

$$\frac{\partial}{\partial t} \rho = -\frac{i}{\hbar} [H, \rho]. \quad (1.2)$$

In a first step the force $F(t)$ is set equal to zero. Eq. (1.2) is then transformed into the interaction picture with respect to $H_0 + H_B$ and solved in a perturbation expansion in $H_I$ up to second order [5], [6]. Tracing out the bath’s degrees of freedom one has to deal with correlation functions of the bath, which are simplified by a Markov assumption, in which memory effects are ignored. Virtual (i.e. energy nonconserving) processes between system and bath are neglected, the interaction is treated in rotating wave approximation. Finally, after the transformation back into the Schrödinger picture and adding $H_F$ to the Hamiltonian, the master equation reads:

$$\frac{\partial}{\partial t} \rho = -\frac{i}{\hbar} [H_0, \rho] + \Lambda(\rho) - \frac{i}{\hbar} [H_F, \rho] \quad (1.3)$$

The operator $\Lambda$, which is linear in $\rho$, contains the gain- and loss- terms of the master equation, the external force $F(t)$ has been added in (1.3) as a pure Hamiltonian component of the motion. This derivation of (1.3) is open to
criticism because it neglects the influence which the driving force should have on the dissipation mechanism. All the harmonics of the driving frequency \( \Omega = 2\pi/T \) contained in the external force \( F(t) \) are expected to couple to the bath modes, whereas the rotating wave approximation leading to (1.3) only takes into account energy differences of \( H_0 \).

In order to incorporate the driving force \( F(t) \) correctly, the authors of [9] made use of the Floquet- theorem, which for the \( T \)-periodic Hamiltonian \( H_0 + H_F \) asserts that there are solutions of the Schrödinger equation of the form

\[
| \Psi_\mu(t) \rangle = e^{i\varphi_\mu t} | U_\mu(t) \rangle
\]

\[
| U_\mu(t + T) \rangle = | U_\mu(t) \rangle
\]

The Floquet- states \( | U_\mu(t) \rangle \) are \( T \)-periodic and will be assumed to form a complete set, in the following. The phases \( \varphi_\mu \) are called “quasi-energies” [8]. The master equation has to be rederived within the Floquet- representation, and now the bath modes couple resonantly at differences of quasi-energies.

The resulting master equation in the interaction picture with respect to \( H_0 + H_F + H_B \) does not contain an extra driving term like (1.3).

In the Floquet- representation a suggestive physical picture seems to be that
the externally forced system is thermalized, and one might believe that the
limiting steady-state would have to be diagonal in the Floquet-basis:

\[ \rho_{st} = \sum_{\mu} P_{\mu} \, |U_{\mu}(t)><U_{\mu}(t)| \]  

(1.5)

The purpose of this article is to examine this assumption. That it cannot be
generally true is easy to see: for example it is clear that the steady-state in a
periodically kicked system with strong damping is more likely diagonal in the
basis of \( H_0 \) than in the Floquet-representation, because the system relaxes
completely between the kicks. We shall clarify this question by the study
of the periodically driven damped harmonic oscillator as a simple system, in
which all calculations can be performed explicitely.

In Section 2. the conventional form (1.3) of the master equation of the driven
oscillator is presented making use of coherent states [10], in which wave pack-
ets or density matrices are expressed in close connection to the corresponding
classical trajectories. This master equation can be solved analytically with
the aid of the Glauber-Sudarshan \( P \)-function [11]. Floquet-states and
quasi-energies of the periodically driven oscillator [14] will be given in Sec-
tion 3., where the representation of coherent states is used again. The density
matrix from Section 2. will be transformed into the Floquet-representation.
The corresponding master equation, though written in the Floquet- basis, is still of the form (1.3). We shall derive the improved master equation in the interaction picture of $H_0 + H_F + H_B$ in Section 4. and compare with the result of Section 3. In this way the limit, in which the correct master equation can be replaced by the conventional approach, will become transparent. From both forms we find that the steady- state is non- diagonal in the Floquet- representation. However, for the oscillator model we are able to find explicitly a basis which diagonalizes the steady- state density matrix $\rho_{(st)}$ for both methods. In Section 5, we show that this basis interpolates between the usual Floquet- states (no damping) and the energy states of the undriven harmonic oscillator (strong damping). In this sense these states can be considered as generalized quasi- energy- states for the master equation. Finally we explain, in which physical situations these new states differ significantly from the usual Floquet- states. Thus the physical picture is recovered, that the driven oscillator expressed in a suitable basis of generalized Floquet- states is simply thermalized by the influence of dissipation.
2. The Conventional Approach

Derivations of the master equation of the harmonic oscillator coupled to a
bosonic heat bath can be found in the literature of quantum optics, from
which [5], [6], [7] are a few representatives. First, we have to specify the
Hamiltonian for the periodically driven harmonic oscillator in interaction
with a heat bath:

\[ H = H_0 + H_B + H_I + H_F \]
\[ H_0 = \hbar \omega_0[a^\dagger a + \frac{1}{2}] \]
\[ H_B = \sum_i \hbar \omega_i b_i^\dagger b_i \]
\[ H_I = \hbar \sum_i (a + a^\dagger)(g_i b_i + g_i^* b_i^\dagger) \]
\[ H_F = -xF(t) \quad ; \quad x = \sqrt{\frac{\hbar}{2m\omega_0}}[a^\dagger + a] \]

\( H_0 \) is the Hamiltonian of an oscillator with frequency \( \omega_0 \), \( H_B \) stands for the
modes of the bath, the interaction \( H_I \) is bilinear in the oscillator- and bath
variables and \( H_F \) is the \( T \)-periodic driving term. The master equation then
reads

\[ \frac{\partial}{\partial t} \rho = -\frac{i}{\hbar} [H_0, \rho] + \pi D(\omega_0) \mid g(\omega_0) \mid^2 \left( ([a, \rho a^\dagger] + [a \rho, a^\dagger])(n_{th}(\omega_0) + 1) + ([a^\dagger, \rho a] + [a^\dagger \rho, a])n_{th}(\omega_0) \right) \]
$$\frac{i}{\hbar} \frac{F(t)}{\sqrt{2}c} (\{\rho, a\} + \{\rho, a\}^\dagger)$$

where $c^2 = m\omega_0/\hbar$ and $n_{th}(\omega_0)$ is the thermal occupation number of the $\omega_0$-bath mode. $D(\omega_0)$ denotes the density of states in the bath at frequency $\omega_0$. The rotating wave approximation has been carried out with respect to energy differences of $H_0$, so only this single frequency is brought into play.

For an oscillator model coherent states $|\alpha>$ usually prove to be the most convenient representation [10], $|n>$ are eigenstates of $H_0$:

$$|\alpha> = e^{-\frac{1}{2}|\alpha|^2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n>$$

$$\alpha = \frac{1}{\sqrt{2}} [cx + i \frac{\bar{h}c}{\hbar} p]$$

$(x,p)$ are values of the classical phase-space variables, over which $|\alpha>$ is centered. This close connection between coherent states and classical trajectories is extremely helpful in the discussion of the solution of (2.2). The density operator expressed in coherent states reads

$$\rho = \int d^2 \alpha P(\alpha, \alpha^*) |\alpha><\alpha|$$

$P(\alpha, \alpha^*)$ is the $P$-function introduced by Glauber and Sudarshan [10], [11].

The diagonal structure of this representation is due to the overcomplete nature of coherent states and must not be misinterpreted in its physical con-
sequences. The great advantage is that the master equation (2.2) for the $P$-function translates into the Fokker-Planck-equation of an Ornstein-Uhlenbeck-process [7]:

$$\frac{\partial}{\partial t} P(\alpha, \alpha^*) = [i\omega_0 + \kappa] \frac{\partial}{\partial \alpha} (\alpha P(\alpha, \alpha^*)) + [-i\omega_0 + \kappa] \frac{\partial}{\partial \alpha^*} (\alpha^* P(\alpha, \alpha^*)) + 2\kappa n_{th} \frac{\partial^2}{\partial \alpha \partial \alpha^*} P(\alpha, \alpha^*)$$

$$+ \frac{i}{\hbar \sqrt{2c}} \left[ \frac{\partial}{\partial \alpha^*} - \frac{\partial}{\partial \alpha} \right] P(\alpha, \alpha^*) \quad (2.5)$$

$$\kappa = \pi D(\omega_0) |g(\omega_0)|^2$$

The solution of (2.5) is

$$P(\alpha, \alpha^*) = \frac{1}{\pi n_{th} (1 - e^{-2\kappa t})} e^{-|\alpha - A(t)|^2/(1 - e^{-2\kappa t})}$$

(2.6)

with the initial conditions

$$P(\alpha, \alpha^*, t = 0) = \delta^{(2)}(\alpha - A(0)),$$

(2.7)

which means that the initial state is a coherent state. $A(t)$ is the complex classical trajectory in the sense of (2.3), which for the damped driven oscillator reads:

$$A(t) = \alpha_0 e^{-(i\omega_0 + \kappa)t} + \alpha_{(st)}(t)$$

(2.8)

$$\alpha_{(st)}(t) = \frac{i}{\hbar \sqrt{2c}} \int_{-\infty}^{t} e^{-(i\omega_0 + \kappa)(t-t')} F(t') dt'$$
In the derivation of the master equation, a small shift of the energies has been ignored in order to be consistent with the weak coupling assumption $\kappa \ll \omega_0$. In this limit, the shifted frequency $\sqrt{\omega_0^2 - \kappa^2}$ is replaced by $\omega_0$, and $\alpha_{(st)}$ denotes the $T$-periodic classical limit-cycle in the complex notation. Finally, the solution of the original master equation (2.2) in eigen-representation of $H_0$ is obtained transforming (2.6):

$$\rho_{m,n} = \int d^2 \alpha < m \mid \alpha > \alpha \mid n > P(\alpha, \alpha^*) \frac{n!}{m!} e^{-\frac{|A(t)|^2}{\sigma + 1}} \frac{\sigma^n}{(\sigma + 1)^{m+1}} A(t)^{m-n} L_n^{m-n} \left(-\frac{|A(t)|^2}{\sigma (\sigma + 1)}\right)$$  \hspace{1cm} (2.9)

$$\sigma = n_{th}(1 - e^{-2\kappa t})$$

This form (2.9) holds for $m \geq n$, $L_n^{\alpha}$ are the generalized Laguerre polynomials [12], [13]. For infinitely long times, the transient parts of $A(t)$ and $\sigma$ vanish leaving $\alpha_{(st)}(t)$ and $n_{th}$, respectively, and the steady-state density matrix $\rho_{(st) m,n}$ is obtained in this limit from (2.9). If the driving force becomes very small such that $|\alpha_{(at)}| \ll n_{th}$, $\rho_{(at) m,n} \equiv (2.9)$ turns into a thermal distribution:

$$\rho_{(at) m,n} = \frac{n_{th}^n}{(n_{th} + 1)^{n+1}} \delta_{m,n}$$  \hspace{1cm} (2.10)

For finite $\alpha_{(at)}$, however, $\rho_{(at) m,n}$ has a more complicated nondiagonal structure.
We will now turn to the question, whether $\rho_{(st)}$ can be diagonalized in the representation of Floquet- states. According to [14], the Floquet- states of the periodically driven harmonic oscillator in position representation are just energy- eigenstates, which move along the classical steady- state trajectory $x_{(st)}(t)$. As we realized in the previous Section 2., it is convenient to express the Floquet- states

$$|\Psi_{\mu}(t)\rangle = e^{i\varphi_{\mu}t} |U_{\mu}(t)\rangle$$  \hspace{1cm} (3.1)

$$|U_{\mu}(t+T)\rangle = |U_{\mu}(t)\rangle$$

in terms of coherent states. The quasi- energy states (3.1) are special solutions of the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\Psi_{\mu}\rangle = [H_0 + H_F] |\Psi_{\mu}\rangle .$$  \hspace{1cm} (3.2)

Expressed in coherent states it turns into a linear partial differential equation for $\langle \alpha | U_{\mu} \rangle$:

$$i \frac{\partial}{\partial t} < \alpha | U_{\mu} > = \left[ \omega_0 \alpha^* - f(t) \right] \frac{\partial}{\partial \alpha^*} < \alpha | U_{\mu} >$$

$$+ \left[ \frac{1}{2} \omega_0 (|\alpha|^2 + 1) - f(t)(\alpha^* + \frac{1}{2}\alpha) + \varphi_{\mu} \right] < \alpha | U_{\mu} >$$

$$f(t) = \frac{1}{\sqrt{2\hbar}} F(t)$$
\( \alpha \) and \( \alpha^* \) are considered to be independent variables. Equation (3.3) can be solved with respect to periodic boundary conditions:

\[
< \alpha | U_\mu(t + T) > = < \alpha | U_\mu(t) > \tag{3.4}
\]

The solutions are quasi-energies and Floquet-states in the representation of coherent states:

\[
< \alpha | U_\mu > = \frac{1}{\sqrt{\mu!}} [\alpha^* - \alpha^0_{(st)}(t)]^\mu e^{i\Xi(t) - \frac{1}{2}|\alpha|^2 - \frac{1}{2}|\alpha^0_{(st)}(t)|^2 + \alpha^* \alpha^0_{(st)}(t)}
\]

\[
\varphi_\mu = -\omega_0(\mu + \frac{1}{2}) + \frac{1}{T} \chi(T) \quad 0 \leq \mu < \infty \tag{3.5}
\]

\[
\chi(t) = \frac{1}{2} \int_0^t f(t')[\alpha^0_{(st)}(t') + \alpha^0_{(st)}(t')]dt'
\]

\[
\Xi(t) = \chi(\cdot : t :) - \frac{t}{T} \chi(T)
\]

\( < \alpha | U_\mu > \) is single-valued in the whole complex plane only if the \( \mu \) are integers. \( : t : \) denotes the time taken modulo the period \( T \); \( \alpha^0_{(st)}(t) \) is the \( T \)-periodic classical trajectory for vanishing dissipation, which might be obtained from \( \alpha_{(st)}(t) \) given in (2.8) in the limit \( \kappa \to 0 \). In position-representation, the states (3.5) are related to shifted energy states [14], which move along the real classical trajectory \( x_{(st)}(t) \). Since (3.5) can be written as

\[
< \alpha | U_\mu > = e^{i\Xi(t) - \frac{1}{2}|\alpha^0_{(st)}(t)|^2} < \alpha | \alpha - \alpha^0_{(st)}(t) > < \alpha - \alpha^0_{(st)}(t) | \mu > \tag{3.6}
\]

13
a similar relation also holds in the representation by coherent states: the
\( \langle \alpha | U_\mu > \) are connected with energy states \( | \mu > \), which move along
the complex classical trajectory \( \alpha^{(s)}(t) \). The quasi-energies \( \varphi_\mu \) and the
additional phase factor \( \Xi(t) \) etc. can be shown to be the same as given
in [14]. The Floquet- states (3.5) enable us to calculate the quasi- energy-
representation of the density operator from the \( P \)- function (2.6) \( (\mu \geq \nu) \):

\[
\tilde{\rho}_{\mu,\nu} = \int d^2 \alpha < \Psi_\mu | \alpha > < \alpha | \Psi_\nu > P(\alpha, \alpha^*)
\]

\[
= e^{i(\varphi_\nu - \varphi_\mu)t} \sqrt{\frac{\sigma^\nu}{\mu^!(\sigma + 1)_{\mu + 1}}} e^{-\frac{|\tilde{A}(t)|^2}{\sigma(\sigma + 1)}} \tilde{A}(t)^{\mu - \nu} L_{\nu}^{\mu - \nu} (-\frac{|\tilde{A}(t)|^2}{\sigma(\sigma + 1)})
\]

\[
\tilde{A}(t) = \alpha_0 e^{-i(\omega_0 + \kappa)t} + \alpha^{(s)}(t) - \alpha^{(s)}(t) \quad (3.7)
\]

\[
\sigma = n_{th}(1 - e^{-2nt}) .
\]

\( \tilde{\rho} \) carries the tilde which denotes the interaction representation with respect
to \( H_0 + H_F \). The main difference between the form (3.7) of \( \tilde{\rho}_{\mu,\nu} \) and \( \rho_{m,n} \)
(2.9) lies in the meaning of the complex trajectory \( \tilde{A}(t) \), which in the steady-
state reduces to

\[
\tilde{A}^{(s)}(t) = \alpha^{(s)}(t) - \alpha^{(s)}(t) \quad (3.8)
\]

Again \( \rho^{(s)} \) is approximately thermalized for a weak driving force \( | \tilde{A}^{(s)}(t) | \ll
n_{th} \), compare eq. (2.9). For a finite driving, however, \( \tilde{A}^{(s)}(t) \) vanishes only
in the conservative case with no dissipation. In the following discussion it is
helpful to consider the master equation itself, of which (3.7) is the solution.

After taking the time-derivative and rearranging the resulting expressions one gets (see Appendix 1 eqs. (A1.7) and (A1.9)):

\[
\frac{\partial}{\partial t} \tilde{\rho}_{\mu,\nu} = \kappa [2\sqrt{(\mu+1)(\nu+1)}\tilde{\rho}_{\mu+1,\nu+1} - (\mu + \nu)\tilde{\rho}_{\mu,\nu}](n_{th} + 1) \\
+ \kappa [2\sqrt{\nu\mu}\tilde{\rho}_{\mu-1,\nu-1} - (\mu + \nu + 2)\tilde{\rho}_{\mu,\nu}]n_{th} \\
+ \kappa [\sqrt{\mu+1}\tilde{\rho}_{\mu+1,\nu} - \sqrt{\nu}\tilde{\rho}_{\mu,\nu-1}]e^{-i\omega_0 t}\alpha_{(st)}^0(t) \\
+ \kappa [\sqrt{\nu+1}\tilde{\rho}_{\mu,\nu+1} - \sqrt{\mu}\tilde{\rho}_{\mu-1,\nu}]e^{i\omega_0 t}\alpha_{(st)}^0(t) 
\]

(3.9)

The first two lines of the above equation have the structure of the master equation for the undriven harmonic oscillator, i.e. only these terms would have appeared if it were sufficient to incorporate the periodic driving just by switching from energy eigenstates to Floquet-states. The last two lines in (3.9), however, in addition contain time-dependent parts, which remain from the external driving mechanism. On the other hand, it is obvious from the parametrization of \(\rho_{(st)}\) in terms of classical trajectories \(\alpha_{(st)}(t)\) and \(\alpha_{(st)}^0(t)\), that the steady-state in general is influenced by dissipation in a way which cannot be fully described by a concept like the usual Floquet-states, which has a physical meaning only for the undamped system.

The master equation (3.9), although expressed in quasi-energy states, has
been derived under the rotating wave approximation based on $H_0$ and assuming that $F(t)$ leaves the dissipative part of the master equation unchanged. The question arises, whether the nondiagonal structure of $\tilde{\rho}(st)$ will survive, if the master equation is rederived directly in the Floquet- representation.

4. The Improved Master Equation

In this section we shall derive the master equation in the Floquet- representation following the lines, which have been worked out in [9] for a Rydberg atom driven by a coherent monochromatic microwave field. The dynamics of $\rho$ is determined by the von Neumann equation (1.2), which, transformed into the interaction picture, allows a weak coupling expansion with respect to $H_I$:

$$\tilde{\rho}(t) - \tilde{\rho}(0) = -\frac{1}{\hbar^2} \int_0^t dt' \int_0^{t'} dt''[[\tilde{\rho}(t''), \tilde{H}_I(t'')], \tilde{H}_I(t')] \quad (4.1)$$

$$\tilde{H}_I(t) = \hbar \sum_i (\tilde{a}(t) + \tilde{a}^\dagger(t))(g_i \tilde{b}_i(t) + g_i^* \tilde{b}_i^\dagger(t))$$

We are interested in the motion of the oscillator variables $\tilde{a}, \tilde{a}^\dagger$ alone, thus we perform the trace over bath variables $\tilde{b}_i, \tilde{b}_i^\dagger$ denoted by $<>_B$. The expectation values of the bath variables read:
\[< \tilde{b}_i(t') \tilde{b}_j(t') >_B = n_{th}(\omega_i) \delta_{i,j} e^{i\omega_i(t'-t')} \]

\[< \tilde{b}_i(t') \tilde{b}_j^\dagger(t') >_B = (n_{th}(\omega_i) + 1) \delta_{i,j} e^{-i\omega_i(t'-t')} \]

Having worked out the commutators and the trace, equation (4.1) becomes:

\[< \tilde{\rho}(t) >_B - < \tilde{\rho}(0) >_B = \int_0^t dt' \int_0^{t'} dt'' \sum_i |g_i|^2 \left[ ([\tilde{a}(t'), \tilde{\rho}(t'') \tilde{a}^\dagger(t'')] + [\tilde{a}^\dagger(t'), \tilde{\rho}(t'') \tilde{a}(t'')] + [\tilde{a}(t'), \tilde{\rho}(t'') \tilde{a}^\dagger(t'')] + [\tilde{a}^\dagger(t'') \tilde{\rho}(t'), \tilde{a}(t')] + [\tilde{a}(t'') \tilde{\rho}(t'), \tilde{a}^\dagger(t')] + [\tilde{a}^\dagger(t'') \tilde{\rho}(t'') \tilde{a}^\dagger(t'')] + (n_{th}(\omega_i) + 1) e^{i\omega_i(t''-t')} + n_{th}(\omega_i) e^{-i\omega_i(t''-t')} \right] \]

We wish to take matrix elements of (4.3) with respect to the time-independent states \(| \Psi_\mu(0) > = | U_\mu(0) >\), which allows us to make use of the selection rules for the matrix elements:

\[< U_\mu(0) | \tilde{a}(t) | U_\nu(0) > = e^{i(\varphi_\nu - \varphi_\mu)t} < U_\mu(t) | a | U_\nu(t) > \]

\[= \sqrt{\nu} e^{-i\omega_0 t} \delta_{\mu+1,\nu} + \alpha_{(st)}^0(t) \delta_{\mu,\nu} \]

and in an analogous way:

\[< U_\mu(0) | \tilde{a}^\dagger(t) | U_\nu(0) > = e^{i(\varphi_\nu - \varphi_\mu)t} < U_\mu(t) | a^\dagger | U_\nu(t) > \]

\[= \sqrt{\nu + 1} e^{i\omega_0 t} \delta_{\mu,\nu+1} + \alpha_{(st)}^0(t) \delta_{\mu,\nu} \]
We shall show the result of this procedure for one commutator as an example:

\[
< U_\mu(0) \mid [\tilde{a}(t'), \tilde{\rho}(t'')\tilde{a}^\dagger(t'')] \mid U_\nu(0) > = (4.6)
\]

\[
\left[ \sqrt{(\mu + 1)(\nu + 1)}\tilde{\rho}_{\mu+1,\nu+1}(t'') - \nu\bar{\tilde{\rho}}_{\mu,\nu}(t'') \right] e^{i\omega_0(t'' - t')}
\]

\[
+ \left[ \sqrt{\mu + 1}\bar{\tilde{\rho}}_{\mu+1,\nu}(t'') - \sqrt{\nu}\tilde{\rho}_{\mu,\nu-1}(t'') \right] e^{-i\omega_0 t'} \alpha_{(st)}^0 s(t'')
\]

The classical complex trajectory may be expanded into a Fourier series:

\[
\alpha_{(st)}^0 (t) = \sum_k \alpha_{(st) k}^0 e^{-ik\Omega t} \tag{4.7}
\]

\[
\alpha_{(st) k}^0 = \frac{f_k}{\omega_0 - k\Omega} \tag{4.8}
\]

The \( f_k \) are Fourier coefficients of the driving force \( f(t) = F(t)/\hbar\sqrt{2c} \). Note that a phase space trajectory has to be passed clockwise and the Fourier coefficients of the positive frequency parts therefore always overcompensate those of the anti-clockwise parts especially for the resonant terms in (4.8).

The \( t'' \)-integration in (4.3) is performed with the aid of the Markov assumption, but still there are oscillating parts in \( t' \). Furtheron we shall drop the rapidly oscillating terms, which contain factors like \( \exp(\pm 2i\omega_0 t') \) and \( \exp(\pm i(\omega_0 + k\Omega)t') \), \( k \geq 0 \); but we shall keep those terms containing \( \exp(\pm i(\omega_0 - k\Omega)t') \), \( k \geq 0 \), because in this case some modes of the driving force might be in resonance with \( \omega_0 \). An exact resonance, however, has to be
avoided, for then the Floquet- description breaks down.

\[
\frac{\partial}{\partial t} \tilde{\rho}_{\mu,\nu} = \kappa [2 \sqrt{(\mu + 1)(\nu + 1)} \tilde{\rho}_{\mu+1,\nu+1} - (\mu + \nu) \tilde{\rho}_{\mu,\nu}] (n_{th} + 1) \\
+ \kappa [2 \sqrt{\mu \nu} \tilde{\rho}_{\mu-1,\nu-1} - (\mu + \nu + 2) \tilde{\rho}_{\mu,\nu}] n_{th} \\
+ \kappa \left[ \sqrt{\mu + 1} \tilde{\rho}_{\mu+1,\nu} - \sqrt{\nu} \tilde{\rho}_{\mu,\nu-1} \right] e^{-i\omega_0 t} \sum_{k \geq 0} \left[ \gamma_k \alpha_{(st)}^0 k + \gamma_{-k} \alpha_{(st)}^0 - k \right] e^{i k \Omega t} \\
+ \kappa \left[ \sqrt{\nu + 1} \tilde{\rho}_{\mu,\nu+1} - \sqrt{\mu} \tilde{\rho}_{\mu-1,\nu} \right] e^{i\omega_0 t} \sum_{k \geq 0} \left[ \gamma_k \alpha_{(st)}^0 k + \gamma_{-k} \alpha_{(st)}^0 - k \right] e^{-i k \Omega t}
\]

in which we introduced

\[
\pi D(\omega_0) \mid g(\omega_0) \mid^2 = \kappa ; \quad \pi D(k\Omega) \mid g(k\Omega) \mid^2 = \kappa \gamma_k
\]

This is an obvious generalization of the “conventional” master equation (3.9), in which the Fourier components of the time- dependent driving terms are weighted according to the coupling with the appropriate bath mode. Both forms become equivalent in the quasistatic limit \( \Omega \to 0 \) and if the distribution of \( \gamma_k \) is broad and unstructured. In this case the negative frequency parts for the strong modes are negligible, and after having completed the remaining expressions in the master equation by some nonresonant terms, the Fourier-modes sum up to \( \alpha_{(st)}^0 \) and \( \alpha_{(st)}^0 \), respectively. Thus (4.9) turns into the conventional form (3.9).

Formally both master equations (3.9) and (4.9) differ only in the explicit time- dependence of the driving terms. In Appendix 1 we demonstrate that
it is possible to find a solution of (4.9) denoted by $\tilde{\rho}_{\mu,\nu}$, which is similar to (3.7), compare eqs. (A1.7), (A1.10):

$$\tilde{\rho}_{\mu,\nu} = e^{i(\varphi_\nu - \varphi_\mu)t} \left[ \frac{\nu!}{\mu! (\sigma + 1)_{\mu + 1}} e^{-\frac{|A(t)|^2}{\sigma + 1}} A(t)^{\mu - \nu} L^{\mu - \nu}_{\nu} \left( -\frac{|A(t)|^2}{\sigma (\sigma + 1)} \right) \right]$$

$$\tilde{\rho}(t) = \alpha_0 e^{-(i\omega_0 + \kappa)t} + \sum_{k \geq 0} \left[ \gamma_k (\alpha_{(st)} k - \alpha^0_{(st)} k) + \gamma_{-k} (\alpha^*_{(st)} - k - \alpha^0_{(st)} - k) \right] e^{-ik\Omega t}$$

$$\sigma = n_{th} (1 - e^{-2\kappa t})$$

Again we observe that the steady-state $\tilde{\rho}_{(st)}$ is non-diagonal in the Floquet-representation, although we used a more accurate derivation of the master equation, which incorporates the external force $F(t)$ in combination with the heat bath correctly.

5. Diagonalization of $\rho_{(st)}$

In the previous sections we have seen, that it is impossible to diagonalize the steady-state density matrix $\rho_{(st)}$ in the Floquet-representation, which consists of solutions of the Schrödinger equation and thus stands for the pure conservative motion without dissipation. The steady-state is affected by the heat bath, and the corresponding density matrix is non-diagonal in a basis of Schrödinger-solutions. The distinction between the conservative
and the dissipative steady-state becomes evident already in the complex classical limit-trajectories \( \alpha_{(st)}(t) \) and \( \alpha^0_{(st)}(t) \). The off-diagonal elements of the version (3.7) of \( \tilde{\rho}_{(st)}^{\mu,\nu} \) depend on the difference between Fourier modes of these trajectories:

\[
\tilde{\rho}_{(st)}^{\mu,\nu}(t) \sim (\alpha_{(st)}(t) - \alpha^0_{(st)}(t))^{[\mu-\nu]}
\]

and a similar statement holds for the improved form (4.11). In this section we shall construct a set of states which diagonalize \( \rho_{(st)} \) exactly. The following argument is given for the form (3.9) of the master equation and its solution (3.7), the generalization for the improved master equation of Section 4. can be achieved replacing \( \alpha_{(st)}(t) \) by a modification, in which the Fourier-modes are weighted with \( \gamma_k \):

\[
\alpha_{(st)}(t) \rightarrow \alpha^0_{(st)}(t)
\]

\[
+ \sum_{k \geq 0} [\gamma_k (\alpha_{(st)} - \alpha^0_{(st)} + \gamma_k (\alpha^*_k - \alpha^0_k) e^{-ik\Omega t}
\]

The quasi-energies and Floquet-states (3.5) are determined by the conservative limit-cycle \( \alpha^0_{(st)}(t) \). It should be a good guess to replace \( \alpha^0_{(st)}(t) \) by \( \alpha_{(st)}(t) \) and to investigate the states:
\begin{align}
|\Psi_\mu(\kappa, t)\rangle &= e^{i\phi_\mu(\kappa)t} |U_\mu(\kappa, t)\rangle \\
<\alpha|U_\mu(\kappa, t)\rangle &= \frac{1}{\sqrt{\mu!}}[\alpha^* - \alpha^*_{(st)}(t)]^\mu e^{i\Xi(\kappa, t) - \frac{1}{2}|\alpha^* - \frac{1}{2}|\alpha_{(st)}(t)|^2 + \alpha^*\alpha_{(st)}(t)} \\
\varphi_\mu(\kappa) &= -\omega_0(\mu + \frac{1}{2}) + \frac{1}{\mu}\chi(\kappa, T) ; \quad 0 \leq \mu < \infty \\
\chi(\kappa, t) &= \frac{1}{2} \int_0^t f(t')[\alpha^*_{(st)}(t') + \alpha_{(st)}(t')] dt' \\
\Xi(\kappa, t) &= \chi(\kappa, :t:) - \frac{t}{T}\chi(\kappa, T)
\end{align}

In the case of a weak external driving force $|\alpha_{(st)}| \ll n_{th}$ the limit- cycle is reduced to the point $\alpha = 0$ and the states (5.3) are oscillator eigenstates:

$$
\lim_{F\to 0} <\alpha|U_\mu(\kappa, t)\rangle = \frac{1}{\sqrt{\mu!}}\alpha^* e^{-\frac{1}{2}|\alpha|^2}
$$

$$
\lim_{F\to 0} (-\hbar\varphi_\mu(\kappa)) = \hbar\omega_0(\mu + \frac{1}{2})
$$

The states $|U_\mu(\kappa, t)\rangle$ interpolate between Floquet- states ($\kappa \to 0$) and oscillator- eigenstates ($F \to 0$), the corresponding phases are quasi- energies for ($\kappa \to 0$) and energies of the harmonic oscillator for ($F \to 0$). In this sense (5.3) can be regarded as a generalization of the Floquet- states for the periodically driven harmonic oscillator in the presence of dissipation. If we express the $P$- function (2.6) in this basis, we simply repeat the procedure used to calculate the Floquet- matrixelements of $\rho$ in Section 3., just $\alpha^0_{(st)}(t)$.
has to be replaced by $\alpha_{(st)}(t)$:

$$\tilde{\rho}_{\mu,\nu}^{(k)} = \sqrt{\frac{\nu!}{\mu!}} e^{i(\varphi_{\nu}-\varphi_{\mu})t} e^{-\frac{|\tilde{A}(t)|^2}{\sigma+1}} \frac{\sigma^\nu}{(\sigma+1)^{\mu+1}} \tilde{A}(t)^{\mu-\nu} L_{\nu}^{\mu-\nu} \left(-\frac{|\tilde{A}(t)|^2}{\sigma(\sigma+1)}\right)$$

$$\tilde{A}(t) = \alpha_0 e^{-(i\omega_0+\kappa)t}$$

$$\sigma = n_{th}(1-e^{-2\kappa t})$$

The corresponding master equation (A1.7), (A1.11) then reads:

$$\frac{\partial}{\partial t} \tilde{\rho}_{\mu,\nu}^{(k)} = \kappa [2\sqrt{(\mu + 1)(\nu + 1)} \tilde{\rho}_{\mu+1,\nu+1}^{(k)} - (\mu + \nu) \tilde{\rho}_{\mu,\nu}^{(k)}] (n_{th} + 1)$$

$$+ \kappa [2\sqrt{\mu\nu} \tilde{\rho}_{\mu-1,\nu-1}^{(k)} - (\mu + \nu + 2) \tilde{\rho}_{\mu,\nu}^{(k)}] n_{th}$$

Equation (5.6) is identical in form to the master equation for the undriven oscillator. We observe from this equation and from its solution (5.5) that the density operator expressed in these generalized Floquet- states relaxes to a thermal distribution (2.10), which is diagonal. The generalized quasi-energy states absorb the influence of the external driving mechanism, and hence they have those properties, which for the undamped system distinguish the Floquet- states [9].

We must point out a possible confusion coming along with the term ”generalized Floquet- states”. The master equation of a periodically driven and damped quantum system like (2.2) can be analysed within the Floquet-
Liouville-supermatrix approach, see e.g. [15]. The density matrix is considered there as a (super-) vector and the master equation is treated like a differential equation with periodic coefficients, thus the Floquet-theorem is applied to the master equation itself. The Floquet-solutions

\[ \rho_{\mu\nu}(t) = e^{-i\Omega_{\mu_0\nu_0}t}\tilde{\rho}_{\mu\nu}(t) \]  

\[ \tilde{\rho}_{\mu\nu}(t + T) = \tilde{\rho}_{\mu\nu}(t) \]

with complex \( \Omega_{\mu_0\nu_0} \) also carry the name "generalized Floquet-states", however, one has to keep in mind that \( \rho_{\mu_0\nu_0}^{\mu_0\nu_0} \) is not a quantum-state but a complete density matrix. One of those with \( \Omega_{\mu_0\nu_0} \) is the steady state \( \rho_{(st)} \), the others have the property \( \text{Tr}[\rho_{\mu_0\nu_0}^{\mu_0\nu_0}] = 0 \) and thus have no physical meaning by itself, but provide a basis in the solution space of the master equation. Moreover the \( \rho_{\mu_0\nu_0}^{\mu_0\nu_0} \) depend on the representation, in which the master equation has been set up. Let us consider our harmonic oscillator problem as an example. The master equation (5.6) is most easily set up within the generalized Floquet-states (5.3) (in our sense). The \( \rho_{\mu_0\nu_0}^{\mu_0\nu_0} \) can be directly computed from an eigen-expansion of corresponding \( P \)-function, see [16]:

24
\[ \rho_{\mu\nu}^{\mu_0\nu_0}(t) = e^{-i\Omega_{\mu_0\nu_0}t}\rho_{\mu\nu}^{\mu_0\nu_0} \]

\[ \Omega_{\mu_0\nu_0} = \omega_0[\mu_0 - \nu_0] - in[\mu_0 + \nu_0] \quad (5.8) \]

\[ \bar{\rho}_{\mu\nu}^{\mu_0\nu_0} = (-1)^{\nu_0}\sqrt{\frac{\mu!}{\nu!}} \left[ \frac{d}{dn_{th}} \right]^{\nu_0} \frac{n_{th}^{\nu_0}}{(n_{th} + 1)^{\mu+1}} \delta_{\mu - \nu, \mu_0 - \nu_0} \]

For the rest of this paper, the term ”generalized Floquet- states” stands for (5.3).

Finally, we shall give a quantitative estimate of the regime of \( \kappa \), in which the generalized Floquet- states are significantly different from the usual ones. Equivalently one might ask, how large \( \kappa \) has to be that the \( T \)- periodic limit-cycle \( \alpha_{(st)}(t) \) with dissipation can be distinguished from the conservative one \( \alpha_{(st)}^0(t) \). The Fourier coefficients of \( \alpha_{(st)}(t) \) are:

\[ \alpha_{(st)} k = i \frac{f_k}{i(\omega_0 - k\Omega) + \kappa} \quad (5.9) \]

For a single mode of (5.9) deviations from the conservative term (4.8) appear, if the inequality

\[ | \omega_0 - k\Omega | \ll \kappa \quad (5.10) \]

is violated for at least one of the strong Fourier components \( k \).

From this discussion we can conclude, that there are physical situations (a
mode near the resonance is considerably excited $k\Omega \sim \omega_0$), in which the
generalized Floquet- states are indeed of importance.

6. Summary and Conclusion

To summarize, we investigated the periodically driven harmonic oscillator
with damping as an example of a quantum system with dissipation influenced
by an external periodic force. Our study revealed, that in general Floquet-
states are not able to incorporate the motion of the periodic steady- state
completly. We have set up and solved the master equation derived in the
conventional approach, in which the influence of the external force on the
coupling with the reservoir is neglected. We compared the result of this
approach with the master equation and its solution, derived directly in the
Floquet- representation, taking full account of the influence of the external
periodic force. From both forms we learned, that the steady- state density
matrix in general is non- diagonal in the Floquet- basis.

However, we found a set of time- dependent states, which are eigenstates of
the steady- state density operator $\rho_{(st)}$. These states depend on the strength
of dissipation and interpolate between quasi- energy states for no damping
and energy states of the undriven oscillator for strong damping. In this
representation \( \rho_{(st)} \) is thermally distributed and the dynamics of the density matrix elements is equivalent to the undriven damped harmonic oscillator. Because of these properties, those interpolating states may be considered as generalized Floquet- states for the dissipative system.

For periodically driven anharmonic oscillators like atoms or molecules in strong monochromatic electromagnetic fields, generalized Floquet- states as defined here by the eigenstates of the steady- state density matrix also exist. In the limit of weak dissipation the diagonalized density operator \( \rho_{(st)} \) should be time- independent at least within a reasonable approximation because it is time- independent in the conservative case. Whether \( \rho_{(st)} \) takes the form of the steady- state of the anharmonic system without periodic driving, like it does for the harmonic oscillator even with arbitrary strength of dissipation, has to be left open for further studies.
Appendix 1 : Derivation of the Master Equation from its Solution

In this appendix we shall show, how the master equation is constructed from the known solution. Let us suppose a density matrix of the form (3.7) ($\mu \geq \nu$, integer):

$$\tilde{\rho}_{\mu,\nu} = e^{i(\varphi_\nu - \varphi_\mu)t} \sqrt{\frac{\nu!}{\mu!}} \frac{\sigma^\nu}{(\sigma + 1)^{\mu + 1}} e^{-\frac{|A(t)|^2}{\sigma + 1}} A(t)^{\mu - \nu} L_\nu^\mu (-\frac{|A(t)|^2}{\sigma(\sigma + 1)}) \sigma = n_{th}(1 - e^{-2\kappa t}). \quad (A1.1)$$

Our purpose is to demonstrate, how the time-dependent complex function $A(t)$ translates into the time-dependent driving terms of the master equations (3.9), (4.9) when $A(t)$ is identified with $\tilde{A}(t)$ or $\ddot{A}(t)$, respectively. We need the derivative of the generalized Laguerre polynomials [12], [13]

$$\frac{d}{dx} L_n^\alpha(x) = -L_{n-1}^{\alpha+1}(x) \quad (A1.2)$$

in order to take the time-derivative $\frac{\partial}{\partial t} \tilde{\rho}_{\mu,\nu}$.
\[ \frac{\partial}{\partial t} \tilde{\rho}_{\mu,\nu} = \left[ i(\phi_{\nu} - \phi_{\mu}) - \frac{\partial}{\partial \tau} \frac{|A|^2}{\sigma + 1} + |A|^2 \frac{\partial}{\partial \tau} \frac{M}{(\sigma + 1)^2} \right] \rho_{\mu,\nu} \]

\[ + e^{i(\phi_{\nu} - \phi_{\mu})} \sqrt{\frac{\nu!}{\mu!} \frac{|A(t)|^2}{\sigma + 1}} \left[ \frac{\nu \sigma - 1}{(\sigma + 1)^{\mu+1}} - \frac{(\mu + 1) \sigma^\nu}{(\sigma + 1)^{\mu+2}} A_{\mu-\nu} \frac{\partial \sigma}{\partial t} L^{\mu-\nu} \left( - \frac{|A|^2}{\sigma \sigma + 1} \right) \right] \]

\[ + \frac{\sigma^\nu}{(\sigma + 1)^{\mu+2}} (\mu - \nu) A_{\mu-\nu} \frac{\partial A}{\partial t} L^{\mu-\nu} \left( - \frac{|A|^2}{\sigma \sigma + 1} \right) \]

\[ + \frac{\sigma^\nu}{(\sigma + 1)^{\mu+1}} A_{\mu-\nu} \left( - \frac{|A|^2}{\sigma \sigma + 1} \right) - |A|^2 (2\sigma + 1) \frac{\partial}{\partial t} \frac{\sigma}{\sigma^2 (\sigma + 1)^2} L^{\mu-\nu+1} \left( - \frac{|A|^2}{\sigma \sigma + 1} \right) \]

\[ \frac{\partial}{\partial \tau} \sigma(t) \text{ is given by} \quad (A1.1) \]

\[ \frac{\partial}{\partial t} \sigma(t) = 2\kappa(n_{th} - \sigma(t)) \quad (A1.4) \]

and \( \frac{\partial}{\partial t} A(t) \) can be written as

\[ \frac{\partial}{\partial t} A(t) = -[i\omega_0 + \kappa]A(t) - \kappa \Gamma(t) , \quad (A1.5) \]

compare (2.8), (3.7) and (4.11). The expressions in (A1.3) can be simplified by means of two further relations [12], [13] of the generalized Laguerre-polynomials

\[ L_{n-1}^\alpha(x) = L_{n}^\alpha(x) - L_{n-1}^\alpha(x) \]

\[ x \frac{d}{dx} L_{n}^\alpha(x) = n L_{n}^\alpha(x) - (n + \alpha) L_{n-1}^\alpha(x) \quad (A1.6) \]
which yields:

\[
\frac{\partial}{\partial t} \tilde{\rho}_{\mu,\nu} = \kappa \left[ 2 \sqrt{\mu + 1} \nu \tilde{\rho}_{\mu+1,\nu-1} - (\mu + \nu) \tilde{\rho}_{\mu,\nu} \right] n_{th} + 1 \\
+ \kappa \left[ 2 \sqrt{\mu} \nu \tilde{\rho}_{\mu-1,\nu-1} - (\mu + \nu + 2) \tilde{\rho}_{\mu,\nu} \right] n_{th} \\
+ \kappa \left[ \sqrt{\mu + 1} \tilde{\rho}_{\mu+1,\nu} - \sqrt{\nu} \tilde{\rho}_{\mu,\nu-1} \right] e^{-i\omega_0 t} \Gamma^*(t) \\
+ \kappa \left[ \sqrt{\nu + 1} \tilde{\rho}_{\mu,\nu+1} - \sqrt{\mu} \tilde{\rho}_{\mu-1,\nu} \right] e^{i\omega_0 t} \Gamma(t) 
\]

We shall add a list of the relevant master equation and solution pairs. The master equation of the conventional approach in the interaction picture of \( H_0 \) expressed in the basis of energy- states of \( H_0 \) is given by (A1.7) with

\[
A(t) = \alpha_0 e^{-(i\omega_0 + \kappa)t} + \alpha_{(st)}(t) \\
\Gamma(t) = -\frac{i}{\kappa h \sqrt{2c}} F(t) .
\]

Its solution, transformed to the Schrödinger picture, is given by (2.9). The corresponding master equation is not given explicitly in the text.

The density matrix (3.7) of the conventional approximation transformed into the basis of Floquet- states, which is the solution of (3.9) is given for

\[
A(t) = \tilde{A}(t) = \alpha_0 e^{-(i\omega_0 + \kappa)t} + \alpha_{(st)}(t) - \alpha_{(st)}^0(t) \\
\Gamma(t) = \alpha_{(st)}^0(t) .
\]
The improved master equation (4.9) and its solution (4.11) are given by (A1.7) with

\[ A(t) = \tilde{A}(t) = \alpha_0 e^{-(i\omega_0 + \kappa)t} \]

\[ \Gamma(t) = \sum_{k \geq 0} \left[ \gamma_k \alpha_{(st)k}^0 + \gamma_{-k} \alpha_{(st)-k}^0 \right] e^{-i k \Omega t} \cdot \]

Finally, the master equation (5.6) and solution (5.5) of the undriven damped oscillator are found, if

\[ A(t) = \alpha_0 e^{-(i\omega_0 + \kappa)t} \]

\[ \Gamma(t) = 0 \].
References


